

# Fokker-Planck models of NGC 6397 – A. The modeling <sup>1</sup>

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## ABSTRACT

This is the first of two papers presenting a detailed examination of Fokker-Planck models for the globular cluster NGC 6397. I show that these models provide a good match to observations of the surface density profile, mass functions at three radii and the velocity dispersion profile. The constraint of requiring the best matches to the mass functions and surface density profiles to occur simultaneously defines a surface in an initial parameter space consisting of the cluster concentration, mass, and limiting radius. I discuss various techniques for locating this surface and the dependence of the quality of the matches on the position of the model on the surface, the initial mass function and the retention rate of neutron stars. The quality of the matches are usually strongly related to the age of the models, but one initial mass function was found for which the quality of the matches are independent of time.

*Subject headings:* globular clusters: individual: NGC 6397 – stellar dynamics

## 1. Introduction

This binary paper is a an outgrowth of previous studies of the dynamics of the globular cluster M71 (Lee, Fahlman, & Richer 1991; Drukier, Fahlman, & Richer 1992, hereafter DFR). In DFR we attempted to compare detailed Fokker-Planck models with observations of M71. The approach taken there was to use star counts to measure the surface density profile and mass function of the cluster and radial velocities to measure the velocity dispersion, and then try to find a Fokker-Planck model to match the observations. For M71 no matching model was found and the nature of the discrepancy suggested that additional physical processes were required in the modeling. One of the main lacks in the DFR models, and in all other detailed comparisons between Fokker-Planck models and observations of globular clusters, was the absence of any allowance for the effects of stellar evolution. The difficulties in the case of M71 left open the

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<sup>1</sup>Postscript figures for this paper are available by anonymous FTP from ftp.ast.cam.ac.uk in the directory /pub/drukier or by email to the author at drukier@mail.ast.cam.ac.uk. This paper has been submitted for publication in *The Astrophysical Journal*.

question of whether these Fokker-Planck models were relevant to the question of globular cluster evolution.

Studies previous to DFR had found models to match observations, but these were for more limited data sets and for clusters with power-law cusps (Grabhorn et al. 1992). Considering this success, it seemed natural to pursue the question of relevance by first adding stellar evolution effects to the model, and then conducting a detailed comparison with a large set of observations of a cusp cluster, in this case NGC 6397.

Since, as will be demonstrated in these papers, matching models can be found, there are two perspectives that can be taken. One perspective is that of the numerical modeler who is concerned with the details of the modeling and the comparison procedure, the size of the initial parameter space, and questions of uniqueness. The second perspective is much more narrowly focused and is concerned with what the models tell us specifically about the current state of affairs in NGC 6397. In order to prevent an entangled perspective I have decided to split the discussion of these two aspects into two separate papers. In this, the first, I will discuss the details of the models used, the fitting procedure and the general results of the modeling. In particular, I will discuss in some detail the effects on the models of changes in the initial parameters. Here the NGC 6397 data will be treated as a guide to the interpretation of the models. In the second paper (Drukier 1994, Paper B) I will look at the results from the other angle by examining the details of the best matching models. The discussions in the two papers are necessarily intertwined and the second especially will refer back to results and diagrams in this paper. The reader might consider them to be an interacting binary.

As it stood in DFR, the Fokker-Planck code, which is descended from the orbit-averaged, isotropic Fokker-Planck code of Cohn (1980), had been extended to include a mass function, a tidal boundary following the formulation of Lee & Ostriker (1987), and a heating term based on the formation and evolution of binaries formed in three-body interactions (“three-body binaries”; Lee 1987; Lee et al. 1991). The models used here have been further extended by introducing the effects of stellar evolution. In DFR, the models started with the mass function as it would be after a Hubble time of stellar evolution. That is, it contained a main sequence terminating at about  $0.8M_{\odot}$  and the degenerate remnants of the higher mass stars. In these models it was assumed that the initial model was at some stage after the massive stars have evolved and that the further evolution of the lower mass stars was unimportant. Such an approach is clearly inconsistent with models meant to follow the full evolution of a globular cluster. The models presented here remove this inconsistency by including stellar evolution and pushing the assumed starting time much earlier conceptually. The expulsion of the left-over gas from the star-formation process is neglected. The details are discussed in the next section.

What is left out of a model can be as important as what is included. These models assume spherical symmetry and an isotropic velocity dispersion. The tidal stripping is idealized by assuming that the tidal boundary is spherically symmetric with respect to the cluster center and

that the strength of the tidal field is constant. The constancy of the tidal field excludes both slow changes and tidal shocks. A globular cluster in our galaxy can certainly be expected to suffer tidal shocks from passages through the disk, passages near the bulge, and from giant molecular clouds within the disk. Weinberg (1994) has recently shown that shock heating can result in large amounts of mass loss for clusters such as NGC 6397. Also excluded are all effects of binaries except for the few “virtual” three-body binaries used as the energy source. Gao et. al (1991) included an initial population of binaries as one component in their two-component Fokker-Planck models. These delay core collapse and leave the post-collapse clusters with fairly large core radii of between 1% and 4% their half-mass radii.

In many ways NGC 6397 is a useful foil to M71. Both lie at about the same distance from the galactic center and the galactic plane, but M71 has the metallicity and kinematics of the disk globular cluster system while NGC 6397 belongs to the halo population. NGC 6397 is also more massive and more centrally concentrated than M71 and is regarded as a post-core-collapse cluster. From isochrone fitting the age of NGC 6397 is  $16 \pm 2.5$  Gyr (Anthony-Twarog, Twarog, & Suntzeff 1992). As discussed in DFR, the dynamical status of M71, ie. whether it is in a pre- or post-collapse phase, is unclear since the models give contradictory indications. NGC 6397, from its high central concentration, is highly evolved dynamically and thus is a good candidate for comparison. The star count data for NGC 6397 is that of Drukier et al. (1993) supplemented by the mass function from Fahlman et al. (1989). The velocity dispersion profile of Meylan & Mayor (1991) has also been used.

I will begin with the description of the numerical models paying special attention to the new feature of stellar evolution. Section 3. will discuss the method used to compare the models with the observations. Since minimal post-facto scaling is possible with these models, the results depend only on the initial parameters and the age of the model. Section 3.2. will define the initial parameters used here and §3.3. will give an overview of the effect varying these has on the resulting model. In total, over 1000 models went into the results to be presented in this paper. They were used to refine the description of the effects of parameter variation on the resulting model fits. I define a model set by their initial mass function (IMF) and the choice of tidal radius (see §3.2.). I will look first at the largest of these model sets, will expand the discussion to include model sets with different tidal radii, and subsequently different IMFs. I will conclude by discussing the implications of these findings for future comparisons. A more general conclusion appears at the end of Paper B.

## 2. Models

With the exception of the inclusion of the effects of stellar evolution, the code used here is basically the same as that discussed in DFR. Briefly, I use the isotropic, orbit-averaged form of the

Fokker-Planck equation, where the distribution function is a function of energy and stellar mass. The clusters are assumed to be spherically symmetric. The coupled Fokker-Planck and Poisson equations are solved by the two step process discussed more fully in Cohn (1980). First the diffusion coefficients are calculated and the distribution function is advanced in time in accordance with the Fokker-Planck equation. At this point the potential and the distribution functions are no longer consistent, so the second step is to solve the Poisson equation subject to the constraint that the distribution function remains the same function of the adiabatic invariant  $q(E)$  in the notation of Cohn(1980). The solution of the Poisson equation is done iteratively.

In order to reverse core collapse, an energy source is required. Here, I estimate statistically the number of binaries formed in three-body encounters and the energy released by each such binary as it is hardened by interactions with the field stars. At any time there are only a few such binaries present in the cluster, which is why the treatment is statistical. The prescription for doing this is discussed in Lee et al. (1991) and DFR. Alternative sources of energy are from initial population of binaries, or from binaries formed by close encounters and subsequent dissipation of orbital energy via tides in their atmospheres. These processes are not included in the models presented here.

The tidal boundary is imposed in energy space by defining the tidal energy boundary as the potential at the radius enclosing a fixed mean density. This radius is referred to as the tidal radius  $r_t$ . The distribution functions are reduced exponentially for energies beyond this boundary with the stripping rate dependent on the difference between the energy and the tidal-energy. The rate is given by the formula in Lee et al. (1991) based on the derivation of Lee & Ostriker (1987). Two modifications have been made here. The first is to remove the discontinuity in the first derivative of the tidal stripping rate with respect to energy. Since the distribution function is a function of energy and not the adiabatic invariant, the iterative solution of the Poisson equation requires that it be regridded in  $E$  to preserve the dependence on  $q(E)$ . The regridding is done via a second-order Taylor expansion of the the distribution function in terms of the adiabatic invariant. The effect of the discontinuity in the stripping rate is to introduce discontinuities in the first derivative of the distribution function. These are then amplified by the regridding procedure and can lead to a catastrophic failure of the Poisson equation solver. To reduce the chances for this, the stripping rate has been smoothed over the transition region using a cubic polynomial which is required to be continuous to the first derivative. This is described in Appendix A.

The second modification concerns the timing of the tidal stripping phase with respect to the two stages of advancing the Fokker-Planck equation and updating the potential. Since we need both the density profile and the potential to define the tidal boundary, the tidal stripping must be done after the potential is updated and is once again consistent with the distribution functions, but before proceeding with the next Fokker-Planck step. This was the method used by Lee & Ostriker (1987). The problem with this is that the stripping is done on the distribution functions and afterward the potential and densities are once again inconsistent with them. If the amount of tidal stripping is small, this is only a small inconsistency; but in the late stages of the model's

evolution the mass becomes small and the tidal losses proportionately large. The mass decreases approximately linearly with time and since  $r_t \propto M^{1/3}$ ,  $\dot{r}_t \propto M^{-2/3}$ . In cases where the rate of decrease in  $r_t$  is large it becomes necessary to find a self-consistent solution. To correct for this problem an iterative scheme for simultaneously doing the tidal stripping and solving the Poisson equation was used. This scheme is described in Appendix B.

In their Fokker-Planck model, Chernoff & Weinberg (1990) used a somewhat different scheme for tidal stripping. They found that in the late stages they were unable to find a self-consistent solution to the Poisson equation and the tidal boundary condition. What I find using the iterative scheme is that a self-consistent solution was possible in these situations. The difference arises in that the tidal boundary condition of Chernoff & Weinberg (1990) is equivalent to  $f(E) = 0$  for  $E > E_t$  (as defined in Appendix A.) which is discontinuous at  $E = E_t$ . The tidal stripping condition used here ensures continuity in  $f(E)$  and its first two derivatives. Thus, a self-consistent solution is still possible even with extreme rates of tidal mass loss.

The addition of the effects of stellar evolution is the main change in the code from DFR. The approach used here is much the same as that used in Chernoff & Weinberg (1990). Even without stellar evolution a mass spectrum is desirable. To introduce this a mass grid is employed, with the mass spectrum being broken into a series of bins, each with its own mass and distribution function. Each of these can be considered a mass species which is meant to represent a range of stars with similar mass. The initial mass for each mass species is taken to be the geometric mean of the masses at the bin boundaries. In order to account for stellar evolution, we simply allow the mass for each mass species to change with time. The simplest way to specify this is to adopt functions for the stellar lifetimes and final masses for stars of a given initial mass. I assume that the stars evolve instantaneously from their initial masses to their final masses without worrying about the details of stellar evolution.

To be more specific, let  $t(m^i)$  be the lifetimes of stars with initial mass  $m^i$  and let  $m^f(m^i)$  be their final masses. I assume that at time  $t(m^i)$  a star with initial mass  $m^i$  becomes a star with mass  $m^f(m^i)$ . The initial mass function (IMF) is given by  $N(m^i)dm^i$ . This is often taken to be a power law

$$N(m) dm \propto m^{-(x+1)} dm. \quad (1)$$

When the mass spectral index (MSI),  $x$ , is defined this way, the Salpeter mass function has  $x = 1.35$ . For a bin  $j$  with boundaries  $m_{j-1}^i$  and  $m_j^i$ ,  $m_{j-1}^i < m_j^i$ , the total mass in the bin is given by

$$M_j = \int_{m_{j-1}^i}^{m_j^i} m^i N(m^i) dm^i. \quad (2)$$

The initial mass of mass species  $j$  is taken to be  $\bar{m}_j^i = \sqrt{m_{j-1}^i m_j^i}$  and the initial number of stars in the bin is  $N_j = M_j / \bar{m}_j^i$ . The final mass for mass species  $j$  is  $\bar{m}_j^f = m^f(\bar{m}_j^i)$ . The mass of species  $j$  is assumed to change linearly from  $\bar{m}_j^i$  to  $\bar{m}_j^f$  over the time interval  $t(m_j^i)$  to  $t(m_{j-1}^i)$ . Since  $t(m^i)$  is a monotonically increasing function,  $t(m_j^i) < t(m_{j-1}^i)$ .

The main effect of mass loss due to stellar evolution is to reduce the depth of the potential and indirectly “heat” the cluster. To remain in virial equilibrium in the shallower potential, the kinetic energy of the cluster stars must also be reduced. Due to the negative specific heat of self-gravitating systems, this results in a net expansion, especially in the core. The closer to the cluster center the mass loss takes place, the more effective is the heating and the stronger the expansion. If, as is done here, the model starts with the relative proportions of the various mass species the same at all radii, then the effectiveness of stellar evolution in causing the expansion will depend on the ratio of the stellar evolution time scale,  $t_{se}$ , to the dynamical evolution time scale. If  $t_{se}$  is long compared to, for example, the central relaxation time, then the most massive stars will have time to sink to the center of the cluster through dynamical friction before they evolve. Their evolution is then a more effective energy source. The practical effect of the stellar evolution mass loss is to expand the cluster and delay core collapse beyond the time expected without such mass loss. The length of the delay, if indeed the mass loss doesn’t destroy the cluster entirely, is strongly dependent on the IMF,  $m^f(m^i)$ ,  $t(m^i)$ , and the initial structure.

Chernoff & Weinberg (1990) based their stellar lifetimes on the Miller & Scalo (1979) compilation for Population I stars. For  $m^i < 4.7M_\odot$  the masses of their white dwarf remnants were based on the formula of Iben & Renzini (1983) with  $\eta = 1/3$ . The intermediate mass stars ( $4.7 < M/M_\odot < 8$ ) were assumed to be completely destroyed in a supernova. Stars with  $M > 8M_\odot$  were assumed to leave a  $1.4M_\odot$  neutron star. For purposes of comparison I ran a model corresponding to a model with a King (1966) model dimensionless central potential  $W_0 = 7$  and  $x = 1.5$  in family 3 of Chernoff & Weinberg. The results of the two models were very similar once differences in the choice of Coulomb logarithm were taken into account.

In terms of finding a model to match NGC 6397, the stellar lifetimes chosen by Chernoff & Weinberg (1990) are not useful because of the effects of metallicity on stellar lifetimes. What is needed are the lifetimes of stars of all masses for  $[Fe/H] = -1.9$  appropriate to NGC 6397. Stellar models of the appropriate metallicity are available for low-mass stars, but no such models have been published for masses above about  $0.95 M_\odot$ . The most extensive set of stellar evolution models are those by the Geneva Observatory group (Schaller et al. 1992; Shaerer et al. 1993a,b; Charbonnel et al. 1993). These cover the mass range from  $0.8$  to  $120 M_\odot$ , but only extend to  $Z = 0.001$ . For stars with the metallicity of NGC 6397, models with  $Z = 0.0002$  are required. To estimate these, the lifetimes until the end of He burning for stars of varying metallicity at constant mass were taken from the Geneva models. For  $Z \leq 0.008$  the lifetimes of stars with  $m > 5M_\odot$  is approximately constant with varying metallicity so the lifetimes for  $Z = 0.001$  were adopted. For  $m < 5M_\odot$  the lifetimes were extrapolated to  $Z = 0.0002$  by a polynomial fit to the lifetimes for  $Z \leq 0.008$ . For  $m < 1.25M_\odot$  the lifetime goes as  $m^{-3.5}$  and lifetimes for stars with  $m < 0.8M_\odot$  have been extrapolated assuming this power law. As a consistency check, I compared these ages with ages derived from the models of Vandenberg (1992). For the  $[Fe/H] = -2.030$  models, the ages agree to within 5%. The lifetimes are given in Table 1.

There are several options for the choice of  $m^f(m^i)$ . First, for  $M > 8M_\odot$  I have assumed that

the remnants are  $1.4M_{\odot}$  neutron stars. The situation for the stars which become white dwarfs is more complicated. Figure 3 of Weidemann and Yuan (1989) shows over a dozen different proposals and models for this relation with a wide range of properties. The effect of varying the  $m^f(m^i)$  relation is to change the total amount of mass lost from the cluster through stellar evolution and to change the rate at which the mass is lost. This is most important in the early stages. A higher mass loss rate and a larger total mass loss at the same rate results in a greater expansion of the cluster. The effect on the time of core-collapse is more complicated. While a large cumulative mass loss results in a more expanded cluster, the total mass is also smaller. This reduces the relaxation time and could cause a quicker collapse. For the models described here I have used the scheme of Wood (1992):

$$m^f(m^i) = 0.4e^{m^i/8}. \quad (3)$$

for  $0.5M_{\odot} < m^i < 8.0M_{\odot}$ . Since stars with masses less than  $0.5 M_{\odot}$  do not evolve until long past the epoch we are interested in, we need not worry about their final masses.

All of the models I discuss in these papers have  $t(m^i)$  based on Table 1 and  $m^f(m^i)$  using eq. (3) for the stars less massive than  $8M_{\odot}$  and  $1.4M_{\odot}$  for the more massive stars.

### 3. Finding a match

Locating a matching model for a particular set of observations is not an easy task. The available parameter space is large and the effects on the resulting models of changing individual parameters are complex and non-linear. The interaction of the various parameters provide tradeoffs which can be played against each other to achieve the desired end, but can also lead to models with quite different initial conditions leading to equally good matches. Experience does lead to some useful guidelines and these will be discussed in §3.3.

#### 3.1. Comparison procedure

To tell how well a particular model matches the NGC 6397, the results of the model must be compared with the observations. Once a set of initial parameters has been decided on, the model is run and its state is periodically saved. Of these data, I have taken those in the interval between 10.5 and 19 Gyr for further analysis. In terms of scaling, once the tidal boundary, the three-body reheating mechanism and stellar evolution have been included in the model there are no global scales left free for adjusting. Thus the comparison procedure is fairly straight forward. The data available for comparison with the models are the surface density profile (SDP) and two mass functions (MFs) from Drukier et al. (1993), the intermediate-distance mass-function from

Fahlman et al. (1989) and the velocity dispersion profile from Meylan & Mayor (1991). I will follow the naming convention from Drukier et al. (1993) and refer to the three mass functions as the du Pont:if, FRST, and du Pont:out MFs in order of distance from the cluster center.

Since the observations are based on the projected distribution of stars in the cluster, the first thing to do is to project the density distributions and velocity dispersions for each mass species in the model. I then simulated the observing procedure by integrating the projected profiles over appropriate regions. For the mass functions, the projected densities for each unevolved mass species were integrated over rectangular regions with the same size and orientation and at the same radial position as the observed fields. The widths of the mass bins were then used to convert the integrated counts into numbers per unit mass. The observed and model mass functions are not on the same grid, so the model mass function is interpolated to give values at the observed mean masses. Since the model MF is smooth this is not difficult. A  $\chi^2$  statistic is calculated for each of the three pairs of observed and model mass functions using the observational uncertainties as weights. The quality of the match is judged by the mean of the three mass function  $\chi^2$  statistics,  $\chi_{MF}^2$ .

Sets of annuli were defined matching the observed radii and mean densities within the annuli were integrated from the model density profile. There is a slight inconsistency here in that many of the observed data points are from sections of annuli rather than full annuli. The mean of  $\chi^2$  from the two magnitude limited surface density profiles,  $\chi_{SDP}^2$ , was used as the figure of merit for the SDP fits.

There are several issues to be addressed before proceeding with the comparison of the model and observed profiles. First, the observed profile is for stars above the main-sequence turn-off. This is because of the brightness of these stars and the high degree of crowding in the images of this concentrated cluster. Therefore, the mass bin to use for comparison is the one which is currently evolving. With the scheme used for implementing stellar evolution, the mass of the currently evolving bin is usually less than the mass of the stars at the turn-off. Once the mass drops, the stars in the evolving mass species becomes less concentrated and the mass species is no longer suitable for comparison. Instead of the evolving species I have used the next less massive one. The interval between 0.74 and 0.90  $M_\odot$  has been divided into seven mass species to ensure that the mass discrepancy is small.<sup>2</sup>

The second issue relates to the widths of the mass bins and the range of masses in the observed surface density profiles. Drukier et al. (1993) produced two surface density profiles with

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<sup>2</sup> For technical reasons, the first grid of models was run twice, with slightly different sets of mass bins. In Figs. 2–4 I will show some contour diagrams giving the results for the binning discussed in the previous paragraph. There was an earlier set of models which only had the finer coverage for five bins with masses from 0.77 to 0.84  $M_\odot$ . (The range was increased to ensure that the mass of the next bin would be close to the turn-off mass for models with ages from 10.3 to 18.5 Gyr.) For both grids of models, the contour plots show the same large-scale features. This demonstrates that the results do not have a strong dependence on the choice of mass bins.



different magnitude limits. The shallower profile ( $I < 14.$ ) extends into the center of the cluster while the deeper profile ( $I < 15.5$ ) is limited to radii greater than  $20''$  from the cluster center. Without detailed information on the mass-luminosity relationship for the observed stars it is very difficult to measure the mass range observed. Therefore, one free scaling parameter was allowed for in comparing the model and observed surface density profiles. When the  $\chi^2_{SDP}$  statistic was calculated a single rescaling was also fit by minimizing the  $\chi^2$  for each of the two SDPs. The two profiles were fit separately and the known offset of a factor of 3.09 (Drukier et al. 1993) was employed. The mean rescaling was adopted and  $\chi^2_{SDP}$  calculated.

Similar techniques could not be employed for the velocity dispersions since the observed velocities were not available. Rather, the projected velocity dispersion profile for the model was plotted together with the observed data points and used to confirm that the time which best fit the surface density profile and the mass functions also matched the dynamical information.

It should be kept in mind during the comparisons discussed below that  $\chi^2_{MF}$  and  $\chi^2_{SDP}$  are independent estimators of the quality of fit. The optimal model will be one that minimizes both at the same time and which also gives a velocity dispersion profile consistent with the data of Meylan & Mayor (1991). The age of the model at the optimal time should also be the age of stars in the cluster. Given the large uncertainties in determinations of the absolute ages of globular clusters, this requirement will not be applied too strictly, but the age should be between 13 and 18 Gyr. As will be seen, once a range of parameters giving good matches is found, locating the best of these becomes a fine tuning problem. In general, I have not tried to find the best-matching age for any given model, but have just adopted the best of the model dumps. While the model could be rerun with finer time resolution, the differences in the fits are small enough to be unimportant given the quality of the data. The models shown in Paper B have been rerun this way at around the age of their best match.

### 3.2. Initial parameters

The IMF I initially used was the same as IMF J in Drukier (1992) where, of the 10 tried, it provided the best fit to the observed NGC 6397 mass functions. In §4.2. I will discuss the effects of variations in the IMF. The mass gridding has been changed to allow for finer gridding between 0.74 and  $0.90 M_\odot$  as discussed above. The IMF is made up of two power-laws, one with mass spectral index 1.5 for  $m < 0.4M_\odot$  and the second with  $x = 0.9$  for  $m > 0.4M_\odot$ . The relative scalings were set so that the mass function is continuous at  $0.4 M_\odot$ . I will refer to models made with this IMF and the set of stellar data in §2. as “U20” models.

The models start as King (1966) models with all species having the same initial profile. The initial structure of the model is defined by four parameters. The first is the dimensionless central potential of the King model,  $W_0$ . The strength of the tidal field is given by the initial tidal radius,

$r_t$  and the initial limiting radius of the model is  $r_l$ . A more useful way to parameterize this is to use the ratio  $r_l/r_t$ . If ratio is unity then the initial model fills its tidal volume, but this need not be the case. If  $r_l/r_t < 1$ , then the model has room to expand before suffering substantial tidal losses. The models are taken to travel on a circular orbit so that the strength of the tidal field is constant. The treatment of tidal stripping is further limited by the assumptions of spherical symmetry and an isotropy velocity dispersion. The tidal radius,  $r_t$ , is given in terms of a cluster with mass  $10^5 M_\odot$ . The initial mass,  $M_0$ , completes the specification of the model once  $r_t$  is rescaled by  $\left(\frac{M_0}{10^5 M_\odot}\right)^{1/3}$  and the initial limiting radius calculated from  $r_l/r_t$ . Unless otherwise specified, I will give  $r_t$  as the value for a  $10^5 M_\odot$  cluster. In this way a value of  $r_t$  can be thought of as specifying the galactocentric distance of the model,  $R_G$ , by assuming a galactic mass model,  $M_G(R_G)$ , and taking  $M_c = 10^5 M_\odot$  in the equation

$$r_t = \frac{2}{3} \left[ \frac{M_c}{2M_G(R_G)} \right]^{1/3} R_G. \quad (4)$$

### 3.3. Guidelines

As can be appreciated from the preceding discussion, the available parameter space is large and a systematic approach is required. To begin with I searched through the  $(W_0, M_0, r_t, r_l/r_t)$  parameter space for an acceptable model keeping the following guidelines and their converses in mind. I refer to this searching stage as the “hunting” mode of running models.

1. In general, an increase in the relaxation time will cause a later core collapse. It also decreases the rate of mass loss through the tidal boundary.
2. Increasing the mass or the limiting radius or reducing  $W_0$  will increase the relaxation time. (An increase in  $W_0$  does not necessarily lead to an earlier core collapse however. If  $W_0$  is large enough, a further increase, by decreasing the central relaxation time, increases the initial mass segregation and the depth of the central potential and increases the amount of expansion due to stellar evolution for the same amount of mass loss. The maximum size reached in the expansion phase can then be larger for a larger  $W_0$  and core collapse is correspondingly delayed.)
3. As  $r_l/r_t$  decreases the tidal mass-loss rate decreases. Thus a decrease in  $r_l/r_t$  gives the result that at any given time the mass and the relaxation time are increased, all else being equal.

Unfortunately, it is difficult to quantify any of these effects as they are also strongly dependent on the IMF. They can be traded off against one another in finding a better fit. As discussed below, the fits to the mass functions and the surface density profile both show well defined minima

as a function of age. What I call a “well-fitting model” will be one where both these minima occur simultaneously. The minima in  $\chi_{SDP}^2$  occur close to, but before, core collapse, so the time of core collapse is a useful marker. The minima in  $\chi_{MF}^2$  cluster around a optimal mass (see §2.1. in Paper B) and can be thought of as occurring when the model mass reaches this value. In Fig. 1 I show  $\chi_{MF}^2$  and  $\chi_{SDP}^2$  as a function of time for one model. The two minima are not aligned and it is desirable to change the parameters to bring them into alignment at, preferably, an age older than 13 Gyr. To achieve this the following rules came in handy for this data set:

- Changing  $M_0$  alone tends to move both the time of optimal mass and the time of core collapse by about the same amount.
- Changing  $W_0$  alone tends to not affect the time of core collapse by very much, but does change the time of optimal mass.
- Increasing  $r_l$  alone (ie.  $r_l/r_t$ ) reduces the time of optimal mass and, to a lesser extent, the time of core collapse.

The inter-relationship of these rules defines the parameter surface containing the good models. For other sets of observations and in other clusters different relationships may apply. In each case, the sensitivity of the results to changes in the initial parameters need to be estimated. They can then be used to formulate similar rules applicable to those data.

## 4. Results

### 4.1. The Parameter Surface

In order to pursue the idea of a lower-dimension surface defined by the well-fitting models I ran a grid of models in the three-dimensional space defined by  $W_0$ ,  $M_0$ , and  $r_l/r_t$ . As I explain in Paper B, the optimal value for  $r_t$  is around 20 pc, but there is a wide range of acceptable values. The well-fitting models found in the hunting stage had  $r_t = 18$  or 19 pc so it was more straight-forward to look for a surface with  $r_t = 18.5$  since points on the surface were already approximately known. I later ran model sets with  $r_t = 17$ , 20, and 21 pc and will discuss them further below. For now, it suffices to note that the choice of  $r_t$  does not affect the results very strongly. The region covered was  $4.01 < W_0 < 6.47$ ,  $3.41 < M_0 < 9.94$ , and  $0.4 < r_l/r_t < 1.10$ .

In Fig. 1 I show  $\chi_{MF}^2$  and  $\chi_{SDP}^2$  as a function of time for a typical model. Note that this is not what I have been calling a “well-fitting” model since the two minima are not coincident. Defining the difference in the time of minima as  $\Delta t \equiv (\text{time in minimum in } \chi_{MF}^2) - (\text{time of minimum in } \chi_{SDP}^2)$ , the locus of well-fitting models is that region of parameter space where  $\Delta t = 0$ . As might

be expected, the well-fitting models define a surface in  $(W_0, M_0, r_l/r_t)$  space. Since only a fraction of the models in the grid happen to lie on this surface, I estimated the position of the surface by interpolating along the grid. I took all the pairs of models differing in only one parameter and with  $\Delta t$  of opposite signs and used linear interpolation between them to find the third parameter. The same procedure for these models, together with the data from the models with  $\Delta t = 0$ , gave the age of the model,  $\chi_{MF}^2$ , and  $\chi_{SDP}^2$  on the  $\Delta t = 0$  surface. This surface is fairly smooth, but has some thickness ( $\pm 0.4$  Gyr) due to the finite time resolution in the model results (see. §5.).

Figure 2 shows this surface of well-fitting models. The contours give the estimated value of  $r_l/r_t$  as a function of  $W_0$  and  $M_0$ . The squares indicate the positions of the estimates used in constructing the contours and the circled squares are models which had  $\Delta t = 0$ . Contouring algorithms generally require points on a regular grid, so for the contour diagrams I defined a grid in  $(W_0, M_0)$  and used bi-linear interpolation to estimate the desired datum at each grid point from the values at the three nearest data points. Any grid point which did not have three data points closer than 3.5 grid spacings were ignored. These ignored points are indicated by dots in Fig. 2 and show the size of the interpolating grid. Features on this scale are artifacts of the contouring process. The contours are spaced by 0.05 in  $r_l/r_t$  with thicker contours every 0.25. The value of  $r_l/r_t$  increases from lower-left to upper-right with the thick contour on the right side being  $r_l/r_t = 1$ . Contours in this diagram were a very reliable guide in estimating the value of  $r_l/r_t$  as a function of the other two parameters.

Figure 3 shows the age of the models on the  $\Delta t = 0$  surface. The contours are at 1 Gyr intervals with the thick lines indicating 12 and 15 Gyr. Models with ages less than 10.5 Gyr and greater than 19 Gyr have been excluded from these contour plots. The upper age limit defines the top-left edge of the contoured region, the lower limit, the lower-edge. Additional good models with higher concentrations probably exist, but some tests with  $W_0 = 9$  models with this IMF indicate that these all core collapse very quickly. An extrapolation from Fig. 2 suggests that high-concentration models would also need to have values of  $r_l/r_t$  much larger than one, ie. we would have to assume that globular clusters start with sizes much larger than the tidal limit imposed at the galactocentric distance of their origin. This is not an unreasonable suggestion. For the high concentrations being considered (eg.  $W_0 > 6$ ), most of the mass is well within the initial tidal boundary. Further, contrary to the assumption here, real globular clusters travel on eccentric orbits and thus feel a time-varying tidal force. If a globular cluster moved closer to the center of the galaxy after its birth, then it would, in effect, be starting its evolution overflowing its tidal boundary. Such high  $W_0$  models will not be discussed here.

Figure 4 show similar contour plots for (a)  $\chi_{MF}^2$ , (b)  $\chi_{SDP}^2$  and (c) their mean. There are several features to note in these diagrams. First, there is a broad region in Fig. 4a where the models match the observed mass functions. The fit improves with higher initial mass at a given initial concentration and the dependence on initial concentration is weak. The models with the lowest initial masses evolve fairly quickly and, if not already excluded for being younger than 10 Gyr, would be excluded for giving a poor match to the MFs. The fits to the mass functions are

quite satisfactory in much of the parameter space.

The match for the surface density profile is more problematic. A comparison of Fig. 4b with Fig. 3 shows that the contours of constant  $\chi_{SDP}^2$  are parallel to the contours of constant age. Further,  $\chi_{SDP}^2$  increases with age and is less than two only for models younger than 14 Gyr. Figure 4c shows the contours of constant  $(\chi_{SDP}^2 + \chi_{MF}^2)/2$  which I use as an overall figure of merit. This mean  $\chi^2$  is dominated by  $\chi_{SDP}^2$ , but at young ages the increase in  $\chi_{MF}^2$  becomes important. The result is a valley in the mean  $\chi^2$  contours where the best models lie. The lower boundary of the valley is truncated in the contour plot by the lower age cutoff in the models, but is readily apparent in the original numbers. This region is occupied by models with ages between 11 and 13 Gyr.

The dependencies of  $\chi_{MF}^2$  and  $\chi_{SDP}^2$  on time implied in Fig. 3 and 4a and b are shown more explicitly in Fig. 5. This plots  $\chi_{MF}^2$  and  $\chi_{SDP}^2$  against the age of the model for all the points defining the  $\Delta t = 0$  surface. Clearly, the best fitting models have an age of about 12 Gyr. If it is assumed that the U20 IMF is correct, then these models would imply that NGC 6397 is 12 Gyr old. This age does contradict the age derived from isochrone fitting ( $16 \pm 2.5$  Gyr, Anthony-Twarog, Twarog, & Suntzeff 1992) and, if it were correct, would suggest a problem either with the stellar evolution models, or with these dynamical models. However, models run with other IMFs (see §4.2.) give either older ages, or no preferred age for NGC 6397. In view of this, the safest thing to do is to reject the assumption that the U20 IMF is correct.

Why is there such a strong dependence on the fit of the SDP with time? As discussed in Cohn (1985) and Chernoff & Weinberg (1990) the central density profile for stars with mass  $m_k$  in this sort of model will have a logarithmic slope

$$\zeta_k = -\frac{d \ln \rho_k}{d \ln r} = \left( 1.89 \frac{m_k}{m_u} + 0.35 \right), \quad (5)$$

where  $m_u$  is the mass of the species dominating the core. Projection effects make the observed surface density flatter by one. Clearly, as the model ages the mass of the stars at the turn-off decreases and it is these stars which are counted for the surface density profile. Given a core dominated by  $1.4M_\odot$  neutron stars, the  $\chi_{SDP}^2$  result requires that the turnoff stars be more massive than  $0.83M_\odot$ . The correlation of  $\chi_{SDP}^2$  with time is a reflection of the dependence of turn-off mass on the age of the model. The correlation would not be changed if a different  $t(m^i)$  relation were used. The only difference would be in the initial parameters needed to produce a well-fitting model of a desired age.

When models are run with a different choice of  $r_t$  the same principle still applies. In Fig. 6(a) to (c) I show contour plots of  $r_l/r_t$ ,  $t$  and the mean  $\chi^2$  for a U20 model set with  $r_t = 20$  pc. The parameter surface is very similar to that for  $r_t = 18.5$  pc, but the models are about 1.6 Gyr older at a given point on the surface. The lines of constant  $\chi_{SDP}^2$  are shifted by a similar amount, but retain the same relationship to the age of the model. Figure 6 (d) to (f) shows a similar series of contour diagrams for  $r_t = 17$  pc. In this case the shift is in the opposite sense, with the models

being about 1.6 Gyr younger than the  $r_t = 18.5$  pc models at a given place on the parameter surface. Again, the  $\chi^2_{SDP}$  contours shift with the age contours.

Column 3 in Fig. 7 show the time dependencies of  $\chi^2_{SDP}$  and  $\chi^2_{MF}$  for four sets of models with IMF U20 and  $r_t$  as indicated in the right margin. The  $r_t = 18.5$  pc panel in this column is based on Fig. 5. (The other of model sets shown in this figure will be discussed in the next section.) The distribution of points with time is a result of the varying coverage of parameter space for each set of models. All four model sets display very similar temporal dependencies although the initial parameters giving rise to a point with a given age are different for each model set.

Since all the well-fitting models have much the same structure in terms of the mass and half-mass radius, it is not surprising that there is very little difference between them in terms of half-mass relaxation time. The number of elapsed half-mass relaxation times,  $\tau \equiv \int \frac{dt}{t_r}$ , also shows little variation amongst all the well-fitting U20 models. Within any set of models  $\tau$  increases by about 15% between 12 and 18 Gyr. Hence, the differences in the SDP matches are not a result of the older models also being significantly more evolved dynamically. This may be reflected in the small differences in the model mass functions, however.

## 4.2. Uniqueness

So far, we have seen that the observations cannot uniquely constrain the initial parameters for NGC 6397, but only a subset of them. Can they constrain the IMF and other stellar data? I have addressed this question by constructing five additional sets of stellar data and then searching for well-fitting models with these sets of data. All of these models retain the same IMF for the mass range 0.1 to  $2.0 M_\odot$ .

One of the five schemes, the “NNS” models assumes that all neutron stars receive a sufficiently large “kick” velocity at their birth to escape the cluster. (Field pulsars are known to have high space velocities [Gunn & Ostriker 1970]. Current estimates suggest that the mean velocity is 450 km s<sup>−1</sup> [Lyne & Lorimer 1994]. Whether these originate due to asymmetries in the collapse or result from the unbinding of binaries is still open to question [qv. Wijers et al. 1992, Bailes 1989].) This model set serves as the alternative limiting case to the retention of all the neutron stars. The other four schemes involve modifications to the IMF. Three change the mass range by extending the lower limit to  $0.05 M_\odot$  (the “L05” models), extending the upper limit to  $30 M_\odot$  (the “U30” models), and restricting the upper limit to  $10 M_\odot$  (the “U10” models). The fifth set of models, the “X2” models have second break in the IMF, this at  $2M_\odot$ , and a mass spectral index  $x = 2$  for more massive stars. (Recently Hill, Madore & Freeman (1994) found a MSI  $x = 2 \pm .5$  for  $m > 9M_\odot$  in a selection of Magellanic Cloud associations. Their study also suggested a somewhat smaller MSI for lower mass stars.) The parameter space was searched in both the hunting mode and by more systematic searches along a grid in the  $(r_t = 18.5 \text{ pc}, W_0, M_0, r_l/r_t)$  parameter space.

U10 and X2 model sets were also run with  $r_t = 20$  pc. Table 2 gives the mass fractions for the various mass components in each IMF. Note that the IMFs are the same in the NNS and U20 IMFs, but that the high mass stars leave no remnants.

For the X2 and NNS models no satisfactory matches were found. In all cases where the models had evolved through core collapse, minima in  $\chi_{SDP}^2$  were seen both before and after core collapse, with a maximum at core collapse. In these models, the mass of the evolving stars is much closer to the mean mass in the core and therefore they show very steep surface density profiles at the time of core collapse (see Fig. 12 in Paper B). The surface density drops off more quickly than observed in the outer region covered by the mass functions. The segregation measure  $S_r(m; \text{du Pont:out}, \text{du Pont:if})$  (defined by eq. (2) in Drukier et al. 1993 as the logarithm of the ratio of the two mass functions) between the du Pont:if and du Pont:out mass functions is much larger than observed indicating that these models suffer too much mass segregation. Coincidences between local minima in  $\chi_{MF}^2$  and  $\chi_{SDP}^2$  occurred in both the collapsing and post-collapse phases. The collapsing models still have large core radii and gave quite poor matches to the surface density profile. For the X2 models, the mass functions were matched better when  $r_t = 20$  pc was used, but there was no improvement to the match of the SDP. Post-core-collapse models have profiles that match the observed SDP fairly well, but these are all older than 18.5 Gyr and the matches to the mass functions are very poor.

Figure 7 summarizes the time dependence of  $\chi_{MF}^2$  and  $\chi_{SDP}^2$  for the model sets with well-fitting models. The U30 model set has much the same time dependency as do the U20 model sets. The best U30 models lie at a somewhat younger age than do the U20 models. This is understandable in terms of the argument given above since the U30 models have a higher proportion of heavy remnants than do the U20 models and the effective  $m_u$  in equation (5) is higher. For the same observed slope at core collapse, the mass of the turn-off stars must be higher and thus younger. The effect is small, but, given the even larger contradiction between the optimal age here, and the isochrone age of NGC 6397, a higher number of neutron stars can be excluded. Panels (a) to (c) of Fig. 8 shows  $r_l/r_t$ , the mean  $\chi^2$ , and age of the models on the  $\Delta t = 0$  surface for the U30 model set. The behavior is very similar to that of the U20 model sets. Note that the shapes of the contoured regions are determined by the range of models runs. There certainly exist well-fitting models beyond these regions, I just haven't looked for them.

That the L05 model set has a later time when the models best fit all the observations is not surprising given the comparison between the U20 model sets and the U30 model set. On the other hand, the dependence of  $\chi_{MF}^2$  on time is much stronger than for the U30 and U20 model sets. It is as strong, though in the opposite sense, as the dependence of  $\chi_{SDP}^2$  on time. The mean  $\chi^2$  is fairly constant with time, but at about 14 Gyr the trade-off between the two is minimized. Panels (d) to (f) of Fig. 8 confirms that the dependence of the mean  $\chi^2$  on time is much weaker than for the U20 or U30 models. There is a large basin of models with  $W_0$  between 4.5 and 6.0 and  $M_0$  between  $4 \times 10^5$  and  $6 \times 10^5 M_\odot$  which give matches of similar overall quality by trading off between  $\chi_{MF}^2$  and  $\chi_{SDP}^2$ .

The behavior of the U10 models serves as a warning against extrapolation. For  $r_t = 18.5$  neither  $\chi^2$  shows any time dependence at all. Further, the mean  $\chi^2$  for this model set is the lowest of any.  $\chi_{SDP}^2$  is at all times as low as that seen in any other model and  $\chi_{MF}^2$  although globally higher than in some other cases, is no higher than its value at the time of minimum mean  $\chi^2$  in the other model sets. An additional model set was run with IMF U10 and  $r_t = 20$  pc. In this model set  $\chi_{SDP}^2$  is still approximately constant with time, and  $\chi_{MF}^2$  now decreases with time; the minimum in the mean is at over 18 Gyr. The difference in the time dependence of  $\chi_{MF}^2$  between the  $r_t = 18.5$  pc and  $r_t = 20$  pc models is consistent with a trend to steeper slope in  $\chi_{MF}^2$  vs.  $r_t$  in the U20 models. Panels (g) to (l) in Fig. 8 make clear the very weak time dependence of the mean  $\chi^2$ .

To further investigate this problem, Fig. 9 presents  $\chi^2$  for each of the three mass functions separately as a function of time for each of the model sets. To reduce the clutter in the diagram I have just plotted the best fitting straight line through each of the sets of estimates. Systematic trends are visible in the U20 column for both the slopes and intercepts of the  $\chi^2$  lines and these trends are also present in the two U10 model sets. The decrease in  $\chi^2$  for the du Pont:out MF with increasing  $r_t$  is understandable as indicating that larger tidal radii are preferred in matching this region of the cluster. On the other hand, the fit to the du Pont:if MF gets worse as  $r_t$  is increased. To expand on the points made at the end of §4.1., the variation in  $\tau$  in any single model set is small and its value does not predict the size of  $\chi_{MF}^2$ . That the variation of  $\chi_{MF}^2$  is so small in the U10 model with  $r_t = 18.5$  is a result of the tradeoff between the three mass functions. There is no correlation between  $\tau$  and  $\chi_{MF}^2$  between data sets since the degree of mass segregation with both time and position also depends on the IMF and  $r_t$ . The detailed reasons for these trends are unclear, but relate to the more general question of mass segregation. This matter warrants further study.

For the U10 model sets  $\chi_{SDP}^2$  is constant for both of the magnitude limited SDPs and does not result from a tradeoff between them. Rather, the improved behavior can be understood in terms of eq.(5). For the U10 model sets, the mean mass in the core at the time of best fit is about  $1.0M_\odot$ , while for the other IMFs it is about  $1.2M_\odot$ . The observed slope of the central SDP is -0.9 (Drukier et al. 1993), giving  $\zeta_k = 1.9$ . Taking  $m_u$  to be the central mean mass, the mass for the observable stars which gives the best match to the SDP is  $0.8M_\odot$  for the U10 model sets and  $1.0M_\odot$  for the others. The rate of change  $\frac{d\zeta_k}{dm_k}$  is larger for the U10 model sets, but the stars of optimal mass reach the turn off during the 12 to 18 Gyr window. As a result,  $\zeta_k$  remains within  $\pm 0.1$  of the observed value during the entire interval. For the other model sets  $\zeta_k$  starts off lower than the observed value and decreases with time. Thus the model fits deteriorate substantially as the age of the model increases.

There are several angles from which to consider the strong age dependence of the model fits. Models with poor matches to either the SDP or the MFs cannot be considered good models of NGC 6397. Similarly, models with ages significantly different from the isochrone age of the cluster must also be considered to be in difficulty. Since the quality of match to the SDP depends on the



mass of the turn-off stars at a given time, the discrepancy could suggest that there are problems with the stellar modeling. However, the existence of the U10 models which do not show the strong time effect, shows that this is not the case and that model sets having their best models at the wrong age probably have the wrong IMF. A somewhat philosophical question remains. Is an IMF giving a model set without any age dependence (such as U10) to be preferred to one with an age-dependent quality-of-fit, but with a preferred age consistent with the isochrone age (such as L05)? If they are not to be preferred, then can we use the existence of a preferred age to learn anything about particular clusters?

### 4.3. Robustness

Up until now the discussion of the matches between the models and the observations has been conducted at a level removed from the actual matches. In this section I will show two matches in order to bring some meaning to the values and differences in  $\chi^2$ . More such matches are shown in Paper B where the purpose is to extract information about NGC 6397.

I will begin with a model which has been selected for having the lowest mean  $\chi^2$  of the well-fitting models with ages between 15 and 17 Gyr. It is the third best of all the well-fitting models, and has a mean  $\chi^2$  only 0.06 larger than the best model. As well, it happens to have the lowest  $\chi_{SDP}^2$  of all the well-fitting models. This model, designated t074, is a U10 model, with  $W_0 = 6.00$ ,  $M_0 = 4.5 \times 10^5 M_\odot$ ,  $r_t = 20$  pc, and  $r_l/r_t = 1.08$ . At the displayed time (Fig. 10) its age is 15.8 Gyr,  $\chi_{MF}^2 = 1.54$ , and  $\chi_{SDP}^2 = 1.06$ . As might be expected, the fit to the SDP is quite good. The mass function matches are not as much of a success since many of the details in the observed MFs are not matched. Of more concern, the du Pont:out MF is systematically higher than the model MF at that radius. This is probably an effect of the choice of  $r_t$ . The other concern is that the model velocity dispersion is systematically lower, although still consistent with, the observed velocity dispersion data. Overall, this is certainly an acceptable model and demonstrates the validity of the Fokker-Planck modeling.

By way of comparison, in Fig. 11 I show the well-fitting model with an age between 15 and 17 Gyr and with the lowest  $\chi_{MF}^2$ . This is the model with the second best MF fit and is only 0.02 worse in  $\chi_{MF}^2$  than the best model. It is also has the fourth-highest  $\chi_{SDP}^2$ . This model, designated GG057, is a U20 model, with  $W_0 = 5.35$ ,  $M_0 = 7.56 \times 10^5 M_\odot$ ,  $r_t = 18.5$  pc, and  $r_l/r_t = 0.78$ . At the displayed time its age is 17 Gyr,  $\chi_{MF}^2 = 1.06$ , and  $\chi_{SDP}^2 = 3.16$ . The central part of the SDP is conspicuously poor. Model GG057 also has more stars in the outer region than model t074. This is not a result of the rescaling of the model SDP (see §3.1.), but is a real effect. That this is so can be seen in the mass functions; their radial positions in the cluster are indicated by the vertical lines in the SDP panel. The MFs for the low-mass stars are almost identical for both models; the differences lie for stars more massive than about  $0.3M_\odot$ . For all three MFs, model

GG057 has more of these stars than does model t074 and the size of the difference increases with radius. The velocity data is matched very well.

## 5. Discussion

I have run eleven sets of models in attempting to match a set of observations of the globular cluster NGC 6397. For any given IMF and galactocentric distance (parameterized as a fiducial tidal radius  $r_t$ ) any two of the remaining parameters  $W_0$ ,  $M_0$ , and  $r_l/r_t$  are independent; the requirement to match all the observations at the same time fixes the third. For most of the IMFs tested the quality of the match to the observations is time dependent with only a fairly narrow interval in which both the surface density profile and the mass functions are matched well. The size of the time dependencies is determined by the IMF and, to a lesser extent, by  $r_t$ . In two model sets with one IMF, only a weak time dependence was seen. While this may still not be the optimal choice of IMF, as things stand the existence of preferred times in the other models cannot be used to constrain the age of NGC 6397 in the face of an IMF without a preferred age. What can be said is that the existence and quality of matching models can put limits on the existence and numbers of stars with masses outside the observed range. A small fraction of neutron stars is required, but not too many. As well, the existence of a very large number of low mass stars also appears unlikely. These, and other constraints relating specifically to NGC 6397, are discussed in more detail in Paper B.

The generalized rules discussed in §3.3. can be put on a firmer footing using the results of the various model sets. The  $(W_0, M_0, r_l/r_t)$  surfaces are curved, but rough estimates can be made of the relationship between changes in the parameters and the times of best match to the SDP or the MFs. Series of models varying in only one parameter can be taken from the grids and used to estimate the variation in the times as a function of the parameters. For the eight models sets shown in Fig. 7 the slopes estimated this way are shown in Table 3. These numbers are meant to be representative and suggest the range of variation possible with variations in the IMF. Changes in  $r_l/r_t$  are about three times more effective in changing the time of optimal mass than the time of core collapse. Changes in  $W_0$  do not effect the time of core collapse all that much. The slopes can also be used to quantify the thickness of the surface. Since the models are checked intermittently even the well-fitting models may not have  $\Delta t = 0$  if they are rerun and checked more frequently. From my list of  $\Delta t$  values, the most common one other than zero is 0.4, suggesting that this is the typical time interval between data saves in the vicinity of well-fitting models. The final three columns of Table 3 give the variations in  $W_0$ ,  $M_0$ , and  $r_l/r_t$  which change  $\Delta t$  by 0.4 Gyr. A model with a single one of these parameters changed by the indicated amount should still be well-fitting.

One thing that is clear from this work is the strong effect the IMF has on the quality of the model fits. Additional data can only serve to further limit the range of initial parameters which

match the observations. In retrospect, it may have been more fruitful to treat the three mass functions as independent constraints rather than to use the combined results. As a first attempt at such an extensive comparison, the more limited goal of matching the ensemble of mass functions simplified the analysis of the results. The velocity data does not give as strong a constraint on individual models as does the SDP and the MFs. The model velocity profiles are much the same for all well-fitting models in a model set. The velocities do provide a stronger limit on the IMF, but the present data set is not precise enough to make firm statements.

These models still do not include all the effects that are expected to affect globular clusters. One such effect is disk shocking. Weinberg (1994) has included this in his Fokker-Planck code and shown that in the inner part of the galaxy, clusters can lose substantial amounts of mass through disk shocking. This has much the same effect as stellar evolution mass loss, but takes place for the entire lifetime of the cluster not just the initial Gyr. The extra mass loss would allow for models with higher initial masses to reach core collapse at the present, but also requires high initial concentrations to prevent them from disrupting entirely. In Fig. 2, only the models on the right edge of the diagram (those with  $W_0 > 6.5$ ) would survive based on Weinberg’s preliminary results. For more distant clusters the initial concentration can be lower.

The techniques I have used here could certainly be extended to other clusters, provided sufficiently detailed sets of observation exists. Once an IMF which can give an good match to those observed has been found, and I have not addressed this question here, it should be fairly straightforward to locate the range of good models, assuming they exist. It is difficult to give explicit rules, even those as rough as the ones given in §3.3., which would apply in all cases, but a little experience with a given data set soon provides these. The surfaces of well-fitting models for the last model sets calculated were located much more quickly than the first ones. No hunting phase was required. One technique is to take a single cut, varying only one of  $W_0$ ,  $M_0$ , or  $r_l/r_t$ . Once the  $\Delta t = 0$  point has been found, simple extrapolations along a regular grid, following the rules in §3.3., quickly find additional extrapolations. After several intersection points have been located, estimates can be made of the shape of the surface. Farther-range extrapolations are often quite successful and only a minimum of non-useful models need to be run. Until this has been tested on other data sets, it is impossible to say how universally this will apply. For NGC 6397, at least, these Fokker-Planck models have been quite successful in matching the observations. The information which can be extracted from these matches is the subject of the accompanying Paper B.

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## A. A smooth tidal boundary

Assume that the cluster moves in a circular orbit at a distance  $R_G$  from the center of the galaxy and that the mass of the galaxy within  $R_G$  is  $M_G$ . From Lee & Ostriker (1987) the tidal stripping is given by

$$\frac{\partial f(E, t)}{\partial t} = -C_t f(E, t) b(E) t_t^{-1}, \quad (\text{A1})$$

where the stripping rate  $b(E)$  is given by

$$b(E) = \begin{cases} [1 - (E/E_t)^3]^{1/2} & E < E_t, \\ 0. & E \geq E_t, \end{cases} \quad (\text{A2})$$

and

$$t_t = \frac{2\pi}{\sqrt{\frac{4\pi}{3} G \rho_t}} \quad (\text{A3})$$

is the orbital periods of the cluster about the galaxy. The tidal-energy boundary  $E_t$  is given by the potential at the radius which encloses a mean density equal to the tidal density  $\rho_t$ , a constant dependent on  $R_G$  and the ratio of the initial mass of the cluster to  $M_G$ . Note that the potential is defined to be positive here with  $\phi \rightarrow 0$  as  $r \rightarrow \infty$ .  $C_t$  is a dimensionless constant giving the overall rate of mass loss per orbital period and is taken to be unity. The discontinuity in the first derivative of the stripping rate is obvious in eq. (A2).

Let

$$\beta = 1 - \left( \frac{E}{E_t} \right)^3. \quad (\text{A4})$$

From eq. (A2),  $b(\beta) = \sqrt{\beta}$  for  $\beta > 0$  and is identically zero for  $\beta < 0$ . For the region  $|\beta| \leq \epsilon$ , where  $\epsilon$  is small, I find a cubic polynomial which has the same values and first derivatives as  $b(\beta)$  at  $\beta = \pm\epsilon$ . The required polynomial is

$$b_1(\beta) = \frac{\sqrt{\epsilon}}{8} \left[ -\left( \frac{\beta}{\epsilon} \right)^3 + \left( \frac{\beta}{\epsilon} \right)^2 + 5\frac{\beta}{\epsilon} + 3 \right]. \quad (\text{A5})$$

For  $|\beta| \leq \epsilon$  eq. (A5) was used instead of eq. (A2) with  $\epsilon$  chosen such that two or three of the energy grid points would fall within  $|\beta| \leq \epsilon$ .

## B. Iterative scheme to ensure self-consistency of the tidal boundary

Following the completion of the Fokker-Planck step, the distribution function, the phase space functions  $p$  and  $q$  (qv. Cohn 1980), and the old potential are stored. I'll refer to this group of functions as model  $\mathbf{F}$  and this initial model as  $\mathbf{F}^0$ . One iteration of the Poisson solver is done to estimate the tidal radius,  $r_t^0$ , and the tidal energy,  $E_t^0$ . The Poisson solver changes  $\mathbf{F}$ , so  $\mathbf{F}$  is reset to  $\mathbf{F}^0$ . Tidal stripping is now done using  $E_t^0$  and a full solution of the Poisson equation

is performed yielding model  $\mathbf{F}^1$ . From  $\mathbf{F}^1$ ,  $r_t^1$  is calculated. If  $r_t^1 = r_t^0$ , to sufficient precision, then model  $\mathbf{F}^1$  is self-consistent both between the distribution functions and the potential (as a result of the Poisson solver) and with respect to the tidal boundary. If  $r_t^1 \neq r_t^0$ , then  $\mathbf{F}$  is reset to  $\mathbf{F}^0$ , stripped using  $E_t^1$  based on  $r_t^1$  and the potential in model  $\mathbf{F}^1$  and fed once again through the Poisson solver. This procedure is repeated until  $r_t^n = r_t^{n-1}$  to sufficient precision (I used a fractional difference of less than  $10^{-3}$ ).

Solving Poisson’s equation, even without this iterative scheme, is the most computer intensive part of the code so it is inefficient to do too many iterations of the tidal stripping. If more than one full solution of Poisson’s equation was required to get  $r_t$  at a particular time step, then the next time step was restricted to be no more than one-half the time step just used. It was found that this procedure worked well and could track the tidal radius in the late stages of evolution even for very small masses.

Table 1: Stellar lifetimes of stars with  $Z = 2 \times 10^{-4}$ .

Mass ( $M_{\odot}$ )	Lifetime (yr)
0.8	$1.56 \times 10^{10}$
0.9	$1.03 \times 10^{10}$
1	$7.02 \times 10^9$
1.25	$3.23 \times 10^9$
1.5	$1.97 \times 10^9$
1.7	$1.41 \times 10^9$
2	$1.01 \times 10^9$
2.5	$5.70 \times 10^8$
3	$3.38 \times 10^8$
4	$1.63 \times 10^8$
5	$9.98 \times 10^7$
7	$4.99 \times 10^7$
9	$3.15 \times 10^7$
12	$1.98 \times 10^7$
15	$1.45 \times 10^7$
20	$1.02 \times 10^7$
25	$7.84 \times 10^6$
40	$5.34 \times 10^6$

Table 2: IMF bin masses (in  $M_\odot$ ) and mass fractions.

L05		U10		U20		U30		X2	
$\bar{m}_j^i$	$M_j$	$\bar{m}_j^i$	$M_j$	$\bar{m}_j^i$	$M_j$	$\bar{m}_j^i$	$M_j$	$\bar{m}_j^i$	$M_j$
0.0825	0.264	0.113	0.0811	0.113	0.0692	0.113	0.0635	0.113	0.0978
0.174	0.0882	0.145	0.0716	0.145	0.0612	0.145	0.0561	0.145	0.0864
0.262	0.0480	0.187	0.0632	0.187	0.0540	0.187	0.0495	0.187	0.0763
0.349	0.0312	0.240	0.0557	0.240	0.0476	0.240	0.0437	0.240	0.0673
0.436	0.0236	0.308	0.0492	0.308	0.0420	0.308	0.0386	0.308	0.0594
0.523	0.0200	0.396	0.0441	0.396	0.0376	0.396	0.0345	0.396	0.0532
0.609	0.0175	0.508	0.0442	0.508	0.0377	0.508	0.0346	0.508	0.0533
0.696	0.0155	0.653	0.0453	0.653	0.0387	0.653	0.0355	0.653	0.0547
0.750	0.00353	0.750	0.00514	0.750	0.00439	0.750	0.00403	0.750	0.00620
0.772	0.00354	0.772	0.00516	0.772	0.00440	0.772	0.00404	0.772	0.00622
0.794	0.00355	0.794	0.00517	0.794	0.00441	0.794	0.00405	0.794	0.00624
0.816	0.00355	0.816	0.00517	0.816	0.00442	0.816	0.00405	0.816	0.00625
0.839	0.00356	0.839	0.00518	0.839	0.00442	0.839	0.00406	0.839	0.00625
0.863	0.00359	0.863	0.00522	0.863	0.00446	0.863	0.00409	0.863	0.00630
0.887	0.00358	0.887	0.00522	0.887	0.00446	0.887	0.00409	0.887	0.00630
0.940	0.0111	0.940	0.0161	0.940	0.0138	0.940	0.0126	0.940	0.0195
1.04	0.0151	1.04	0.0220	1.04	0.0187	1.04	0.0172	1.04	0.0265
1.21	0.0238	1.21	0.0347	1.21	0.0296	1.21	0.0272	1.21	0.0418
1.62	0.0568	1.62	0.0828	1.62	0.0707	1.62	0.0648	1.62	0.0999
2.83	0.0998	2.53	0.0975	2.83	0.124	2.83	0.114	2.83	0.122
4.90	0.0617	3.75	0.0687	4.90	0.0767	4.90	0.0704	4.90	0.0408
6.93	0.0453	4.96	0.0535	6.93	0.0564	6.93	0.0517	6.93	0.0204
9.80	0.0661	6.17	0.0440	9.80	0.0823	11.1	0.122	9.80	0.0204
13.9	0.0486	7.38	0.0375	13.9	0.0604	18.6	0.0776	13.9	0.0102
17.9	0.0386	8.94	0.0525	17.9	0.0481	26.1	0.0575	17.9	0.00611

Table 3: Representative relations between the time of best match and the model parameters.

Model set <sup>a</sup>		$\frac{dt_{MF}}{dW_0}$	$\frac{dt_{MF}}{dM_0}$ <sup>b</sup>	$\frac{dt_{MF}}{d(r_l/r_t)}$	$\frac{dt_{SDP}}{dW_0}$	$\frac{dt_{SDP}}{dM_0}$ <sup>b</sup>	$\frac{dt_{SDP}}{d(r_l/r_t)}$	$\Delta W_0$ <sup>c</sup>	$\Delta M_0$ <sup>b,c</sup>	$\Delta r_l/r_t$ <sup>c</sup>
U30	18.5	4.4	2.3	-39	0.0	1.8	-12	0.09	0.8	0.01
U20	17.	4.2	2.3	-30	0.6	1.8	-9	0.1	0.8	0.02
U20	18.5	6.0	2.7	-34	1.0	1.9	-12	0.08	0.6	0.02
U20	20.	6.5	2.8	-36	2.8	2.1	-15	0.09	0.6	0.02
U20	21.	5.1	2.9	-27	0.7	2.3	-7	0.09	0.7	0.02
L05	18.5	7.2	3.0	-38	0.9	2.2	-13	0.06	0.4	0.02
U10	18.5	5.9	4.0	-34	1.4	2.7	-15	0.09	0.3	0.02
U10	20.	4.0	4.3	-34	-0.7	3.0	-7	0.09	0.3	0.02

<sup>a</sup>Stellar data and  $r_t$  in parsecs.

<sup>b</sup>Mass measured in units of  $10^5 M_\odot$ .

<sup>c</sup>Change in parameter required to change  $\Delta t$  by 0.4 Gyr.



## REFERENCES

- Anthony-Twarog B.J., Twarog, B.A., & Suntzeff, N.B. 1992, AJ, 103, 1264
- Bailes, M. 1989, ApJ, 342, 917
- Charbonnel C., Meynet G., Maeder A., Schaller G., Schaerer D. 1993, A&AS, 101, 415
- Chernoff, D.F. & Weinberg, M.D. 1990, ApJ, 351, 121
- Cohn, H. 1980, ApJ, 242, 765
- Cohn, H. 1985, in IAU Symp. 113, The Dynamics of Star Clusters, ed. J. Goodman & P. Hut (Dordrecht: Reidel), 161
- Drukier, G. A. 1992, Ph.D. Thesis, University of British Columbia
- Drukier, G. A. 1994, co-submission (Paper B)
- Drukier, G.A., Fahlman, G.G., & Richer, H.B. 1992, ApJ, 386, 106, 2335 (DFR)
- Drukier, G.A., Fahlman, G.G., Richer, H.B., Searle, L. & Thompson, I. B. 1993, AJ, 106, 2335
- Fahlman, G.G., Richer, H.B., Searle, L. & Thompson, I. B. 1989, ApJ, 343, L49
- Gao, B., Goodman, J., Cohn, H. & Murphy, B. 1991, ApJ, 370, 567
- Grabhorn, R.P., Cohn, H.N., Lugger, P.M., & Murphy, B.W. 1992, ApJ, 392, 86
- Gunn, J.R. & Ostriker, J.P. 1970, ApJ, 160, 979
- Hill, R.J., Madore, B.F., & Freeman, W.L, 1994, ApJ, 429, 204
- Iben, I. Jr. & Renzini, A. 1983, ARA&A, 21, 271
- King, I. 1966, AJ, 71, 64
- Lee, H.M. 1987, ApJ, 319, 801
- Lee, H.M. & Ostriker, J.P. 1987, ApJ, 322, 123
- Lee, H.M., Fahlman, G.G., & Richer, H.B. 1991, ApJ, 366, 455
- Lyne, A.G. & Lorimer, D.R. 1994, Nature, 369, 127
- Meylan, G. & Mayor, M. 1991, A&A, 250, 113
- Miller, G.E. & Scalo, J.M. 1979, ApJS, 41, 513
- Schaller G., Schaerer D., Meynet G., & Maeder A. 1992, A&AS, 96, 269
- Schaerer D., Meynet G., Maeder A. & Schaller G. 1993a, A&AS, 98, 523
- Schaerer D., Charbonnel C., Meynet G., Maeder A. & Schaller G. 1993b, A&AS, 102, 339
- VandenBerg, D. A. 1992, ApJ, 391, 685
- Weidemann, V. & Yuan, J. W. 1989, in IAU Collq. 114, White Dwarfs, ed. G. Wegner (Berlin: Springer-Verlag), 1

Weinberg, M.D. 1994, preprint

Wijers, R.A.M.J., van Paradijs, J., & van den Heuvel, E.P.J. 1992, A&A, 261, 145

Wood, M.A. 1992, ApJ, 386, 539

Fig. 1.— Plot of  $\chi_{MF}^2$  (dashed) and  $\chi_{SDP}^2$  (solid) against model age. The vertical dash marks the time of core collapse. The time of optimal mass is at about 14.5 Gyr. This is not a well-fitting model since the times of minima for the two  $\chi^2$  statistics are not coincident. The discontinuities occur at the times of the data saves.

Fig. 2.— Contour diagram of  $r_l/r_t$  plotted against  $W_0$  and  $M_0$  for the U20,  $r_t = 18.5$ pc model set. The contours are based on the estimates of the position of the  $\Delta t = 0$  surface indicated by the squares. The circled squares are well-fitting models. The dots outside the contoured region indicate the size of the grid used to construct the contours. Features on this scale and smaller should be ignored. The contours are spaced every 0.05, with heavier contours every 0.25. The  $r_l/r_t = 0.5$  and  $r_l/r_t = 1.0$  contours are labeled. Model GG057 (see §4.3. and Fig. 11) is the circled square at  $(W_0, M_0) = (5.35, 7.56)$ .

Fig. 3.— As Fig. 2 but for the ages of the models on the  $\Delta t = 0$  surface. The contours are spaced every Gyr and the 12 and 15 Gyr contours are labeled. Time increase from lower-right to upper-left. The age range for results to be included in the figures is 10.5 to 19 Gyr and the isochrone age of NGC 6397 is  $16 \pm 2.5$  Gyr. For clarity, the positions of the model estimates have not been repeated and the non-contoured region has been left blank.

Fig. 4.— As Fig. 3 for (a)  $\chi_{MF}^2$ , (b)  $\chi_{SDP}^2$ , and (c)  $(\chi_{MF}^2 + \chi_{SDP}^2)/2$ . In each diagram the contours are spaced by 0.1 and the heavier contours every 0.5. Some of the heavier contours are labeled. In (a)  $\chi_{MF}^2$  decreases for older models (cf Fig. 3) and in (b)  $\chi_{SDP}^2$  increases with age. The models with the best overall  $\chi^2$  cluster along the lower edge of (c) with the valley referred to in the text lying between the  $\chi^2 = 1.5$  contours.

Fig. 5.— Minimal  $\chi_{MF}^2$  (triangles) and  $\chi_{SDP}^2$  (circles) vs. time for the U20 model set with  $r_t = 18.5$ pc. The open symbols are minima from actual runs, the filled symbols are estimates of the value of  $\chi^2$  on the  $\Delta t = 0$  surface.

Fig. 6.— Contour diagrams of the  $\Delta t = 0$  surface for the U20 model sets with (a)–(c)  $r_t = 20$ pc and (d)–(f)  $r_t = 17$ pc. The symbols have the same meaning as in Fig. 2 except that the dotted line indicates the boundary of the tested region. (a) & (c)  $r_l/r_t$ . The contours are spaced every 0.05. This corresponds to Fig. 2. (b) & (e) Age. The contours are spaced every 1 Gyr. This corresponds to Fig. 3. (c) & (f) Mean  $\chi^2$ . The contours are spaced every 0.1. This corresponds to Fig. 4c. The heavy contours are as labeled and the direction of increase is indicated by the arrow.

Fig. 7.— Minimum  $\chi_{MF}^2$  (filled squares) and  $\chi_{SDP}^2$  (open squares) vs. time as estimated on the  $\Delta t = 0$  surface. The columns are labeled by their IMF and the rows by the value in pc of  $r_t$  used for the model set. The U20,  $r_t = 18.5$  pc panel repeats the solid symbols in Fig. 5. For the model sets where  $\chi^2$  varies with time, the best models are those lying near the intersection of the two  $\chi^2$  locii. The U10,  $r_t = 18.5$  pc model set gives the most consistently good models independent of age.

Fig. 8.— As Fig. 6 for (a)–(c) the U30,  $r_t = 18.5$  pc model set; (d)–(f) the L05,  $r_t = 18.5$  pc model set; (g)–(i) the U10,  $r_t = 18.5$  pc model set; and (j)–(l) the U10,  $r_t = 20.$  pc model set. Model t074 (Fig. 10) is the circled square at  $(W_0, M_0) = (6.0, 4.5)$  in panel (j). The contour spacing is as in Fig. 6. The heavy contours have the indicated values. For each model set, the first panel corresponds to Fig. 2, the second to Fig. 3 and the third to Fig. 4c. The plotted functions increase in the direction of the arrow except that in (f) the  $\chi^2 = 1.9$  contours surround regions of minima with  $\chi^2$  increasing towards the outer edge of the contoured area, and in (i)  $\chi^2$  is constant over most of the contoured region, increasing slightly at the lower boundary.

Fig. 9.— This figure shows the values of  $\chi^2$  vs. time on the  $\Delta t = 0$  surface for each of the three observed mass functions and their mean ( $\chi_{MF}^2$ ). The model sets are the same as in Fig. 7. One can see in the U20 column systematic trends with  $r_t$  and these are repeated qualitatively for the two U10 model sets. The details of the behavior of the matches to the observed mass functions are a strong function of the IMF.

Fig. 10.— Comparison between the NGC 6397 observations and model t074 in the U10,  $r_t = 20$  pc model set. Clockwise from upper left: The surface density profile; the FRST mass function; (top) the du Pont:if mass function and (bottom) the du Pont:out mass function; and the velocity dispersion profile. The dashed line in the mass function panels indicates the shape of the IMF.

Fig. 11.— As Fig. 10 for model GG057. The dotted lines in the SDP panel (upper left) indicate the radial positions of the three MFs; in order of increasing distance from the cluster center:du Pont:if, FRST, and du Pont:out.